Conductance distribution at criticality: 
one-dimensional Anderson model with random long-range hopping

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We study numerically the conductance distribution function \( w(T) \) for the one-dimensional Anderson model with random long-range hopping described by the Power-law Banded Random Matrix model at criticality. We concentrate on the case of two single-channel leads attached to the system. We observe a smooth transition from localized to delocalized behavior in the conductance distribution by increasing \( b \), the effective bandwidth of the model. Also, for \( b < 1 \) we show that \( w(\ln T/T_{\text{typ}}) \) is scale invariant, where \( T_{\text{typ}} = \exp\langle \ln T \rangle \) is the typical value of \( T \). Moreover, we find that for \( T < T_{\text{typ}} \), \( w(\ln T/T_{\text{typ}}) \) shows a universal behavior proportional to \( (T/T_{\text{typ}})^{-1/2} \).

The study of transport properties of systems at the Anderson metal-insulator transition (MIT) has been a subject of intensive research activity for several decades. In particular, the interest has been focused on the conductance \( T \) and its corresponding probability distribution \( w(T) \) [1–8]. At the MIT, \( w(T) \) has been found to be universal, i.e. size independent, but dependent on the given model, dimensionality, symmetry, and even boundary conditions of the system. \( w(T) \) has been studied for systems in two and more dimensions with a large number of attached single-channel leads. In fact, concerning \( T \) and \( w(T) \), the regime of small number of leads and the case of one-dimensional (1D) systems have been left almost unexplored [8–10]. In the present work we study numerically \( w(T) \) for a 1D system at the MIT described by the Power-law Banded Random Matrix (PBRM) model at criticality with two single-channel leads attached to it (the minimal configuration needed to measure \( T \)).

The PBRM model [11, 12] at criticality describes 1D tight-binding samples with random long-range hopping of length \( L \) represented by \( L \times L \) real matrices whose elements are statistically independent random variables drawn from a normal distribution with zero mean, \( \langle |H_{ij}|^2 \rangle = 0 \), and a variance decaying as a power law \( \langle |H_{ij}|^2 \rangle \sim \langle |i - j|/b \rangle^2 \), where \( b \) is a parameter. There are two prescriptions for the variance of the PBRM model: the so-called non-periodic,

\[
\langle |H_{ij}|^2 \rangle = \frac{1}{2} \frac{1 + \delta_{ij}}{1 + (|i - j|/b)^2},
\]

(1)

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where the 1D sample is in a line geometry; and the periodic,
\[
\langle |H_{ij}|^2 \rangle = \frac{1}{2} \frac{1 + \delta_{ij}}{1 + [\sin (\pi |i - j|/L) / (\pi b/L)]^2},
\]
where the sample is in a ring geometry. Field-theoretical considerations [11–13] and detailed numerical investigations [12, 14, 15] verified that the PBRM model shows all the key features of the Anderson MIT at the critical point. Thus the PBRM model possesses a line of critical points \( \hat{b} \in (0, \infty) \).

Using standard methods [16, 17] we attach two semi-infinite single-channel leads to the 1D sample described by the non-periodic and the periodic versions of the PBRM model at criticality. Then, we calculate the \( 2 \times 2 \) scattering matrix in the standard form
\[
S(E) = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}.
\]
The leads are attached to the first two sites of the sample. That is, in the non-periodic version of the PBRM model, Eq. (2), we attach the leads at the boundary of the system. While in the periodic version, Eq. (1), we attach them to the bulk. In the latter case, finite size effects are considerably reduced. However, the quantities we analyze below are \( L \)-independent once \( L \) is much larger than the number of attached leads for both versions of the PBRM model.

Once the scattering matrix is calculated we can compute the dimensionless conductance \( T = \text{Tr}(t^t) \) and its probability distribution \( w(T) \). Notice that for \( b \to \infty \), \( H \) reproduces the Gaussian Orthogonal Ensemble. Then, in that limit we expect to recover the Random Matrix Theory (RMT) prediction for \( w(T) \) [18]:
\[
w(T)_{\text{RMT}} = \frac{1}{2\sqrt{T}},
\]
which is valid in the absence of direct processes.

In the following we focus on \( w(T) \) for the PBRM model at criticality. We will compare RMT predictions (at least for large enough \( b \)) with numerical simulations, where the statistics is collected by sampling over different disorder realizations. All quantities reported here were obtained for \( L = 50 \) with \( 10^6 \) ensemble realizations. Moreover, we have verified that our results are invariant when increasing \( L \).

For \( b < 1 \), \( w(T) \) is highly concentrated close to \( T = 0 \). So it is more convenient to analyze \( w(\ln T) \) instead. In Fig. 1(Left)(a) we show \( w(\ln T) \) for several values of \( b \). Notice that the distribution functions \( w(\ln T) \) do not change their shape or width by varying \( b \), thus being scale invariant. In fact, \( \langle \ln T \rangle \) clearly displays a linear behavior when plotted as a function of \( \ln b \), see Fig. 1(Left)(b). Then, all distributions \( w(\ln T) \) fall one on top of the other when shifting them along the \( x \)-axis by the typical value of \( T \), \( T_{\text{Typ}} = \exp(\langle \ln T \rangle) \propto b^2 \), as shown in Fig. 1(Left)(c). In Fig. 1(Right) we present the same quantities as in Fig. 1(Left) but now for the non-periodic version of the PBRM model at criticality. We observe very similar results in both cases.

Notice that for \( T < T_{\text{Typ}} \), \( w(\ln T) \) is proportional to \( (T/T_{\text{Typ}})^{-1/2} \), see panels (c) in Fig. 1. Moreover, we found that this behavior is universal for the PBRM model as can be seen in Fig. 2, where we plotted the cumulative distribution \( w_c(x) = \int_x^\infty w(x')dx' \), \( x = \ln T/T_{\text{Typ}} \), for several small and large values of \( b \).

In Fig. 3 we show \( w(T) \) for large \( b \) (\( b \geq 0.4 \)) for both, the periodic and non-periodic versions of the PBRM model at criticality. In the limit \( b \to \infty \), \( w(T) \) is expected to approach the RMT prediction of Eq. (3). However, once \( b \geq 4 \), \( w(T) \) is already well described by \( w(T)_{\text{RMT}} \). Moreover, we have found that \( w(T) \propto T^\mu \) for \( T \in [0.5, 1] \), where, according to the RMT prediction, \( \mu \to -0.5 \) as \( b \to \infty \). In Fig. 4(a) we plot \( \mu \) as a function of \( b \) as measured from the histograms of Fig. 3. We noticed that for \( b \gtrsim 1 \), \( \mu \) is proportional to \( b^{-2} \). This observation together with the analytical estimation for the correlation dimension of the eigenfunctions [12] \( D_2(b) = 1 - (2\pi b)^{-1} \), \( b \gg 1 \), allowed us to conclude that \( \mu(D_2) \propto (1 - D_2)^2 \).
Fig. 1 (online colour at: www.ann-phys.org) \textbf{Left} \cite{A} \textbf{Right}: (a) Conductance probability distribution $\ln w(\ln T)$ for the periodic [non-periodic] PBRM model at criticality for several values of $b < 1$. (b) $\langle \ln T \rangle$ as a function of $b$ (symbols). The black dashed line is the best fit of the data for $b < 1$ to the logarithmic function $A + \ln b^2$, with $A \approx -1.35$. \cite{A}. (c) $\ln w(\ln T)$ for $b < 1$ scaled to $T_{typ} = \exp(\langle \ln T \rangle) \sim b^2$. The black dashed line has slope 1/2 and is plotted to guide the eye.

Fig. 2 (online colour at: www.ann-phys.org) \textbf{Left} \cite{B} \textbf{Right}: Cumulative conductance distribution $\ln w_c(\ln T/T_{typ})$ for the periodic [non-periodic] PBRM model at criticality for several values of $b$. The black dashed line has slope 1/2 and is plotted to guide the eye.
for $D_2$ approaching unity, as can be seen in Fig. 4(b). Thus, $\mu(D_2)$ provides the possibility of evaluating $D_2$ by means of scattering experiments.

To conclude, we observed a transition from localized- to delocalized-like behavior in the conductance distribution of the PBRM model by moving $b$ from small ($b \ll 1$) to large ($b > 1$) values. For small $b$, we showed that $w(\ln T)$ is scale invariant with the typical value of $T$, $T_{\text{typ}}$, as scaling factor. Also, for $T < T_{\text{typ}}$ and irrespective of $b$, we found the universal behavior $w(\ln T/T_{\text{typ}}) \sim (T/T_{\text{typ}})^{-1/2}$. Finally, we showed that the RMT limit, expected for $b \to \infty$, is already recovered for relatively small values of $b$: $b \geq 4$. Our conclusions are valid for leads attached to the bulk of the system as well as for leads attached at the boundary.

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